**Stochastic differential equations with non-Lipschitz coefficients: II. Dependence with respect to initial values**

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**Abstract .** The existence of the unique strong solution for a class of stochastic differential equations with non-Lipschitz coefficients was established recently. In this paper, we shall investigate the dependence with respect to the initial values. We shall prove that the non confluence of solutions holds under our general conditions. To obtain a continuous version, the modulus of continuity of coefficients is assumed to be less than $|x-y|\log\dfrac{1}{|x-y|}$. In this case, it will give rise to a flow of homeomorphisms if the coefficients are compactly supported.

## 0. Introduction

Let $\sigma: R^d \to R^d\otimes R^m$ and $b: R^d \to R^d$ be continuous functions. It is well known (see [IW], [RY]) that the following Itô s.d.e:

$$(0.1)\qquad dX(t)=\sigma(X(t))\,dW_t+b(X(t))\,dt,\quad X(0)=x_o$$

has a weak solution up to a lifetime $\zeta$. It has been proved recently in [FZ2] (see [FZ1] for a short version) that the s.d.e. (0.1) has a unique strong solution if the coefficients $\sigma$ and $b$ satisfy the following non-Lipschitz conditions

$$(H1)\qquad \begin{cases} ||\sigma(x)-\sigma(y)||^2 & \leq & C\,|x-y|^2\,r(|x-y|^2), \\ |b(x)-b(y)| & \leq & C\,|x-y|\,r(|x-y|^2) \end{cases}$$

for $|x-y|\leq\delta_o$, where $r:]0,\delta_o]\to\mathbf{R}_+$ is a strictly positive function of class $C^1$ satisfying (i) $\lim_{s\to 0} r(s)=+\infty$, (ii) $\displaystyle\int_0^{\delta_o}\frac{ds}{sr(s)}=+\infty$ and (iii) $\displaystyle\lim_{s\to 0}\frac{sr'(s)}{r(s)}=0$.

Let $X_t(x_o,w)$ be the solution of the s.d.e. (0.1). It is well-known that the solution admits a continuous version $\tilde{X}_t(x_o,w)$ if the coefficients $\sigma$ and $b$ are locally Lipschitzian (see [Pr]), and it gives rise to a flow of homeomorphisms if the coefficients are globally Lipschitzian (see [Ku]). We refer also to [Ma2], [IW] and [El] for the study of stochastic flow

of diffeomorphisms, to [Ma1] for the non-Lipschitz feature in the study of homeomorphisms of the circle $S^1$. Many interesting phenomena for stochastic differential equations with non-Lipschitz coefficients have been elucidated in [LJR1,2]. In this work, we shall investigate the dependence of solutions with respect to the initial values under the non-Lipschitz condition $(H1)$, which can not be covered in [LJR1,2] and generalize [Ma1] to general situation.

The organization of the paper is as follows. In section 1, we shall prove that the non confluence (or non-contact) property holds under $(H1)$. The function $r$ considered in $(H1)$ includes obviously all functions like $\log\frac{1}{\xi}$, $\log\frac{1}{\xi}\log\log\frac{1}{\xi}, \cdots$. Such kind of properties were also studied by M. Emery in an early work [Em] for Lipschitz case, and by T.Yamada and Y. Ogura for a non-Lipschitz case in [YO]. The conditions in [YO] includes also the function $\xi\log\frac{1}{\xi}$, but their mixed condition about $\sigma$ and $b$

$$\lim_{s\downarrow 0}\frac{\sup_{s\leq t\leq\delta_o}\Big[\kappa(t)\int_t^{\delta_o}du/\rho^2(u)\Big]}{\int_s^{\delta_o}\Big[\int_t^{\delta_o}du/\rho^2(u)\Big]dt}=0$$

where $\rho^2(u)=u^2r(u^2)$ and $\kappa(u)=ur(u^2)$, is not easy to be checked in general. In section 2, we shall establish a continuous version of the solutions. The main tool for doing this is the Kolmogorov's modification theorem, which will work in the case where $r(\xi)=\log\frac{1}{\xi}$. However, it seems difficult to apply the modification theorem in the case where $r(\xi)=\log\frac{1}{\xi}\log\log\frac{1}{\xi}$. Finally in section 3, we shall prove that the continuous version of solutions will give rise to a flow of homeomorphisms if the coefficients are compactly supported.

## 1. Non contact property

**Theorem 1.1** *Assume (H1) and the s.d.e* (0.1) *has no explosion. If $x_o\neq y_o$, then $X_t(x_o,w)\neq X_t(y_o,w)$ almost surely for all $t>0$.*

**Proof.** Consider

$$\psi(\xi)=\int_\xi^1\frac{ds}{sr(s)}\quad\text{and}\quad\Phi(\xi)=e^{\psi(\xi)},\quad\text{for }1\geq\xi>0.$$

We have

$$\Phi'(\xi)=-\frac{\Phi(\xi)}{\xi\,r(\xi)}\leq 0.$$

By condition (i) and (iii) about the function $r$, it exists $1>\delta>0$ and a constant $C>0$ such that

$$\Phi''(\xi) = \frac{\Phi(\xi)\,\big(1 + r(\xi) + \xi r'(\xi)\big)}{(\xi r(\xi))^2} \le C\Phi(\xi)\frac{1}{\xi^2 r(\xi)} \tag{1.1}$$

for $0 < \xi < \delta$. To be simplified, denote $X_t(x_o) = X_t(x_o, w)$. Let $\eta_t = X_t(x_o) - X_t(y_o)$ and $\xi_t(w) = |\eta_t(w)|^2$. We have

$$\begin{aligned} d\xi_t =& 2\big\langle \eta_t, \big(\sigma(X_t(x_o)) - \sigma(X_t(y_o))\big)\, dW_t\big\rangle \\ &+ 2\big\langle \eta_t, b(X_t(x_o)) - b(X_t(y_o))\big\rangle\, dt \\ &+ ||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2\, dt, \end{aligned}$$

and the stochastic contraction $d\xi_t \cdot d\xi_t$ is given by

$$4\big|\big(\sigma^*(X_t(x_o)) - \sigma^*(X_t(y_o))\big)\eta_t\big|^2\, dt$$

where $\sigma^*$ denotes the transpose matrix of $\sigma$. Without loss of generality, we may assume $|x_o - y_o| < \frac{\delta}{2}$. Let $0 < \varepsilon < |x_o - y_o| < \dfrac{\delta}{2}$. Define

$$\tau_\varepsilon = \inf\big\{t > 0, \xi_t \le \varepsilon\big\}, \quad \tau = \inf\big\{t > 0, \xi_t = 0\big\}.$$

We have $\tau_\varepsilon \uparrow \tau$ as $\varepsilon \downarrow 0$. Let

$$\zeta = \inf\big\{t > 0,\ \xi_t \ge \frac{3\delta}{4}\ \big\}.$$

Using Itô formula,

$$\begin{aligned} \Phi\big(\xi_{t\wedge\tau_\varepsilon\wedge\zeta}\big) = \Phi(\xi_o) &+ 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)\big\langle \eta_s, \big(\sigma(X_s(x_o)) - \sigma(X_s(y_o))\big)\, dW_s\big\rangle \\ &+ 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)\big\langle \eta_s, b(X_s(x_o)) - b(X_s(y_o))\big\rangle\, ds \\ &+ \int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi'(\xi_s)||\sigma(X_s(x_o)) - \sigma(X_s(y_o))||^2\, ds \\ &+ 2\int_0^{t\wedge\tau_\varepsilon\wedge\zeta} \Phi''(\xi_s)\big|\big(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o))\big)\eta_s\big|^2\, ds. \end{aligned} \tag{1.2}$$

By $(H1)$ we have

$$\big|\big(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o))\big)\eta_s\big|^2 \le \xi_s^2 r(\xi_s).$$

Combining with (1.1), we get

$$2\Phi''(\xi_s)\big|\big(\sigma^*(X_s(x_o)) - \sigma^*(X_s(y_o))\big)\eta_s\big|^2 \le C\,\Phi(\xi_s).$$

Again by $(H1)$ and the expression of $\Phi'(\xi)$, we have

$$\Big|\Phi'(\xi_s)\big\langle\eta_s, b(X_s(x_o)) - b(X_s(y_o))\big\rangle\Big| \leq C\,\Phi(\xi_s).$$

Therefore according to (1.2),

$$\Phi\big(\xi_{t\wedge\tau_\varepsilon\wedge\zeta}\big) \leq \Phi(\xi_o) + \text{martingale} + C\int_0^{t\wedge\tau_\varepsilon\wedge\zeta}\Phi(\xi_s)\,ds.$$

Taking expectation, we get

$$\begin{aligned}\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta})\Big) &\leq \Phi(\xi_o) + C\mathbb{E}\Big(\int_0^{t\wedge\tau_\varepsilon\wedge\zeta}\Phi(\xi_s)ds\Big)\\ &\leq \Phi(\xi_o) + C\int_0^t \mathbb{E}\big(\Phi(\xi_{s\wedge\tau_\varepsilon\wedge\zeta})\big)\,ds\end{aligned}$$

which implies that

$$\mathbb{E}\Big(\Phi(\xi_{t\wedge\tau_\varepsilon\wedge\zeta})\Big) \leq \Phi(\xi_o)e^{C\,t}, \quad \text{for all } \ t>0.$$

Consequently,

$$P(\tau_\varepsilon < t\wedge\zeta)\Phi(\varepsilon) \leq \Phi(\xi_o)e^{C\,t}, \quad \text{for all } \ t>0.$$

Now first letting $\varepsilon \to 0$, we obtain

$$P(\tau < t\wedge\zeta) = 0, \quad \text{for all } \ t>0,$$

and then letting $t\to\infty$ we get $P(\tau<\zeta)=0$. Therefore, $\xi_.$ is positive almost surely on the interval $[0,\zeta]$. Now define $T_0 := 0$,

$$T_1 := \zeta, \qquad T_2 = \inf\big\{t>0,\ \xi_t \leq \frac{\delta^2}{4}\big\}$$

and generally

$$T_{2n} = \inf\big\{t > T_{2n-1},\ \xi_t \leq \frac{\delta^2}{4}\big\}, \qquad T_{2n+1} = \inf\big\{t>T_{2n},\ \xi_t \geq \frac{3\delta}{4}\big\}$$

Clearly $T_n \to \infty$ almost surely as $n\to\infty$. By definition, $\xi_.$ is positive on the interval $[T_{2n-1}, T_{2n}]$. By pathwise uniqueness of solutions, $X$ enjoys the strong Markovian property. Starting again from $T_{2n}$ and applying the same arguments as in the first part of the proof, one can show that $\xi_.$ is positive almost surely also on the interval $[T_{2n}, T_{2n+1}]$. This completes the proof. ∎

**Theorem 1.2** *Assume that for* $|x|\geq 1$,

$$(H2) \qquad \begin{cases} ||\sigma(x)||^2 & \leq & C\,(|x|^2\,\rho(|x|^2)+1),\\ |b(x)| & \leq & C\,(|x|\,\rho(|x|^2)+1)\end{cases}$$

*where* $\rho : [1,+\infty[\to \mathbf{R}_+$ *is a function of class* $\mathcal{C}^1$ *satisfying (i)* $\displaystyle\int_1^\infty \frac{ds}{s\rho(s)+1} = +\infty$, *(ii)* $\displaystyle\lim_{s\to+\infty} \frac{s\rho'(s)}{\rho(s)} = 0$ *and (iii)* $\displaystyle\lim_{s\to+\infty} \rho(s) = +\infty$. *Then* $\lim_{|x_o|\to+\infty} |X_t(x_o,w)| = +\infty$ *in probability.*

**Proof.** Let $f \in C^1(R_+)$ be a fixed, strictly positive $C^1$ function on $R_+$ that satisfies

$$f(s) = \rho(s) \qquad \text{for} \qquad s \geq 1.$$

Then it is easy to see that $(H2)$ holds for all $x \in R^d$ , with $\rho$ replaced by $f$. From now on, we will use $C$ to denote a generic constant which may change from line to line.
Define

$$\psi(\xi) = \int_0^\xi \frac{ds}{sf(s)+1}, \quad \xi \geq 0$$

and put

$$\Phi(\xi) = e^{-\psi(\xi)}.$$

Keeping the assumptions on $\rho$ in mind it follows that

$$\Phi'(\xi) = -\frac{\Phi(\xi)}{\xi f(\xi)+1} \leq 0,$$

and

$$\begin{aligned} \Phi''(\xi) &= \frac{\Phi(\xi)}{(\xi f(\xi)+1)^2}[1+f(\xi)+\xi f'(\xi)] \\ &\leq C\Phi(\xi)\frac{f(\xi)}{(\xi f(\xi)+1)^2}. \end{aligned} \tag{1.3}$$

Let $\xi(t) = |X_t(x_0)|^2$. For any constant $R > 0$, define

$$\tau_R = \inf\big\{t \geq 0,\ |X_t(x_0)| \leq R\big\}.$$

By Itô formula, we have

$$\begin{aligned} \Phi(\xi_{t\wedge\tau_R}) =& \Phi(|x_0|^2) + 2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\big\langle X_s, \sigma(X_s)dW_s\big\rangle \\ &+2\int_0^{t\wedge\tau_R} \Phi'(\xi_s)\big\langle X_s, b(X_s)\big\rangle\, ds + \int_0^{t\wedge\tau_R} \Phi'(\xi_s)||\sigma(X_s)||^2\, ds \\ &+2\int_0^{t\wedge\tau_R} \Phi''(\xi_s)\,|\sigma^*(X_s)X_s|^2\, ds. \end{aligned} \tag{1.4}$$

By $(H2)$, it holds that

$$\frac{|\sigma^*(X_s)X_s|^2}{(\xi_s\, f(\xi_s)+1)^2} \leq C\ \frac{\xi_s\,(\xi_s\, f(\xi_s)+1)}{(\xi_s\, f(\xi_s)+1)^2}$$

Together with (1.3), we have

$$\int_0^{t\wedge\tau_R} \Phi''(\xi_s)\,|\sigma^*(X_s)X_s|^2\,ds \le C\int_0^{t\wedge\tau_R} \Phi(\xi_s)\,ds.$$

Similarly, we have for some constant $C>0$,

$$\big|\Phi'(\xi_s)\big\langle X_s, b(X_s)\big\rangle\big| \le C\,\Phi(\xi_s), \quad s>0.$$

Combining above inequalities, we get from (1.4)

$$E\big(\Phi(\xi_{t\wedge\tau_R})\big) \le \Phi(|x_0|^2) + C\int_0^t E\big(\Phi(\xi_{s\wedge\tau_R})\big)\,ds,$$

which implies that

$$E\big(\Phi(\xi_{t\wedge\tau_R})\big) \le \Phi(|x_0|^2)\,e^{Ct}.$$

This gives that

$$P\big(\tau_R \le t\big)\Phi(R^2) \le E\big(\Phi(\xi_{t\wedge\tau_R})\big) \le \Phi(|x_0|^2)\,e^{Ct}.$$

Therefore,

$$P\Big(\inf_{0\le s\le t}|X_s(x_o)| \le R\Big) \le e^{Ct}\,\exp\Big\{-\int_{R^2}^{|x_o|^2}\frac{ds}{sf(s)+1}\Big\},$$

which tends to 0 when $|x_o| \to +\infty$. ∎

## 2. Continuous dependence of initial values

In this section, we shall show that the solution to (0.1) admits a version that is jointly continuous in $(t, x_o)$. Let's begin with the following lemma.

**Lemma 2.1** *Let $p \ge 1$. Assume that the coefficients $\sigma$ and $b$ are compactly supported, say,*

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \ge R, \tag{2.1}$$

*and the function $r$ in (H1) $:]0,\delta_o] \to \mathbf{R}_+$ is decreasing. Then there exists a constant $C_p > 0$ such that for $|x| \le R+1$ and $|y| \le R+1$,*

$$\begin{cases} ||\sigma(x)-\sigma(y)||^2 & \le & C_p\,|x-y|^2\,r\Big((\frac{|x-y|^2}{M})^p\Big), \\ |b(x)-b(y)| & \le & C_p\,|x-y|\,r\Big((\frac{|x-y|^2}{M})^p\Big) \end{cases} \tag{2.2}$$

*where* $M = \dfrac{4(R+1)^2}{\delta_o} \ge 1$.

**Proof.** Because of the similarity, we only prove the conclusion for $b$. Let $\delta = \inf\{\frac{\delta_o}{2}, \frac{1}{2}\}$. If $|x-y| \le \delta$, by hypothesis $(H1)$,

$$|b(x) - b(y)| \le C\ |x-y| r(|x-y|^2) \le |x-y|\, r\Big(\big(\frac{|x-y|^2}{M}\big)^p\Big), \tag{2.3}$$

since $r$ is supposed to be decreasing. Remark that

$$\inf_{\delta \le \xi \le 2(R+1)} \xi\, r\Big(\Big[\frac{\xi^2}{M}\Big]^p\Big) \ge \delta\, r\big(\delta^p\big)$$

and $\sup\limits_{x,y} |b(x) - b(y)| \le 2||b||_\infty$. Therefore there exists a constant $C_p > 0$ such that

$$|b(x) - b(y)| \le C_p\, |x-y|\, r\Big(\big(\frac{|x-y|^2}{M}\big)^p\Big) \quad \text{for} \quad |x-y| \ge \delta. \tag{2.4}$$

Combining (2.3) and (2.4), we get the result. ∎

**Lemma 2.2** *Assume the same hypothesis as in lemma 2.1 and furthermore $\xi \to \xi r(\xi)$ is concave over $]0, \delta_o]$. Let $p \ge 1$ be an integer. For $|x_o| \le R+1$ and $|y_o| \le R+1$, set*

$$\eta_t = |X_t(x_o) - X_t(y_o)|^2 \quad \text{and} \quad \xi_t = \big(\frac{\eta_t}{M}\big)^p$$

*where $M$ is the constant appeared in (2.2). Put $\phi(t) = E(\xi_t)$. Then we have*

$$\varphi'(t) \le C_p\, \varphi(t)\, r(\varphi(t)) \tag{2.5}$$

*for some constant $C_p$.*

**Proof.** We remark that under the assumptions (2.1), for any $|x_o| \le R+1$, $|X_t(x_o)| \le R+1$ almost surely for all $t \ge 0$. In fact, this can be seen as follows. Define

$$T = \inf\big\{t \ge 0;\ |X_t(x_o)| \ge R+1\big\}$$

Set $Y_t(x_0) = X_{t\wedge T}(x_o)$. Then

$$Y_t = x_0 + \int_0^{t\wedge T} \sigma(Y_s) dW_s + \int_0^{t\wedge T} b(Y_s) ds.$$

Since

$$I\!\!E\Big(\int_0^t \sigma^2(Y_s)\big(\mathbf{1}_{(s<T)} - 1\big)^2\, ds\Big) = I\!\!E\Big(\int_T^t \sigma^2(Y_s)\, ds\Big) = 0,$$

we have for $t \ge 0$,

$$\int_0^{t\wedge T} \sigma(Y_s) dW_s = \int_0^t \sigma(Y_s) dW_s \quad \text{and} \quad \int_0^{t\wedge T} b(Y_s) ds = \int_0^t b(Y_s) ds,$$

almost surely. We see that $\{Y_t,\ t \geq 0\}$ satisfies the same stochastic differential equation as $\{X_t,\ t \geq 0\}$. By the pathwise uniqueness in [FZ2], we conclude that $Y_t = X_t$ a.s. for all $t \geq 0$, which proves the claim.

Now we shall proceed as in [Fa]. By Itô's formula,

$$\begin{aligned} d\eta_t =& 2\big\langle X_t(x_o) - X_t(y_o), \big(\sigma(X_t(x_o)) - \sigma(X_t(y_o))\big)\, dW_t\big\rangle \\ &+ 2\big\langle X_t(x_o) - X_t(y_o), b(X_t(x_o)) - b(X_t(y_o))\big\rangle\, dt \\ &+ ||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2\, dt, \end{aligned}$$

and

$$d\xi_t = \frac{1}{M^p}\Big(p\eta_t^{p-1} d\eta_t + \frac{1}{2}p(p-1)\eta_t^{p-2} d\eta_t \cdot d\eta_t\Big). \tag{2.6}$$

Now using (2.2),

$$\begin{aligned} &\frac{2p}{M^p}\eta_t^{p-1}\Big|\big\langle X_t(x_o) - X_t(y_o), b(X_t(x_o)) - b(X_t(y_0))\big\rangle\Big| \\ &\leq C_p \frac{2p}{M^p}\eta_t^{p-1}\eta_t\, r(\xi_t) = 2pC_p\xi_t r(\xi_t). \end{aligned}$$

Similarly , we get the same control for other terms in (2.6) except the martingale part. Now by expression of $d\xi_t$, we have

$$\xi_{t+\varepsilon} - \xi_t \leq M_{t+\varepsilon} - M_t + C_p \int_t^{t+\varepsilon} \xi_s r(\xi_s)\, ds$$

where $M_t$ is the martingale part of $\xi_t$. It follows that

$$\mathbb{E}(\xi_{t+\varepsilon}|\mathcal{F}_t) - \xi_t \leq C_p \mathbb{E}\Big(\int_t^{t+\varepsilon} \xi_s r(\xi_s) ds|\mathcal{F}_t\Big)$$

where $\mathcal{F}_t$ is the natural filtration generated by $\{w(s);\ s \leq t\}$. Let $\varphi(t) = \mathbb{E}(\xi_t)$. Then

$$\varphi(t+\varepsilon) - \varphi(t) \leq C_p \int_t^{t+\varepsilon} \mathbb{E}(\xi_s\, r(\xi_s))\, ds.$$

It follows that

$$\varphi'(t) \leq C_p \mathbb{E}(\xi_t\, r(\xi_t)).$$

Since

$$\xi \to \xi r(\xi) \quad \text{is concave over} \quad ]0, \delta],$$

we have

$$\varphi'(t) \leq C_p\, \varphi(t)\, r(\varphi(t)). \tag{2.7}$$

**Theorem 2.3** *Assume (H1) and the s.d.e* (0.1) *has no explosion. Consider* $r(\xi) = \log \frac{1}{\xi}$. *Then there exists a version of* $X_t(w, x_o)$ *such that* $(t, x_o) \to X_t(w, x_o)$ *is continuous over* $[0, +\infty[\times \mathbf{R}^d$ *almost surely.*

**Proof.** It has been proved in [FZ2] that the s.d.e (0.1) has no explosion under the hypothesis $(H2)$. We split the proof into two steps.
**Step 1**. Assume that $\sigma$ and $b$ are compactly supported, say,

$$\sigma(x) = 0 \quad \text{and} \quad b(x) = 0 \quad \text{for } |x| \geq R.$$

Let $\varphi$ be defined as in Lemma 2.2. Solving (2.7), we get $\varphi(t) \leq (\varphi(0))^{e^{-C_p t}}$ or explicitly

$$\mathbb{E}\Big(|X_t(x_o) - X_t(y_o)|^{2p}\Big) \leq C_p\, |x_o - y_o|^{2pe^{-C_p t}}.$$

On the other hand, it is easy to see that

$$\mathbb{E}\Big(|X_t(x_o) - X_s(x_o)|^{2p}\Big) \leq C_p |t - s|^p.$$

Therefore,

$$\mathbb{E}\Big(|X_t(x_o) - X_s(y_o)|^{2p}\Big) \leq C_p \Big[|t - s|^p + |x_o - y_o|^{2pe^{-C_p t}}\Big]. \tag{2.8}$$

Fix $p > d+1$. Choose a constant $T_o > 0$ small enough such that $2pe^{-C_p T_0} > d+1$. It follows from (2.8) and Kolmogorov's modification theorem that there exists a version of $X_t(w, x_0)$, denoted by $\tilde{X}_t(w, x_0)$, such that $(t, x_o) \to \tilde{X}_t(w, x_o)$ is continuous over $[0, T_o] \times \{|x_o| \leq R + 1\}$ almost surely. But

$$X_t(x_o, w) = x_o \quad \text{if} \quad |x_o| > R.$$

We conclude that $(t, x_o) \to \tilde{X}_t(x_o, w)$ can be extended continuously to $[0, T_o] \times \mathbf{R}^d$. Let $(\theta_{T_o} w)(t) = w(t + T_o) - w(T_o)$. Define for $0 < t \leq T_o$,

$$\tilde{X}_{T_o + t}(x_o, w) = \tilde{X}_t\big(\tilde{X}_{T_o}(x_o, w), \theta_{T_o} w\big).$$

Then $\tilde{X}_{T_o + \cdot}(x_o, w)$ satisfies the s.d.e (0.1) driven by the Brownian motion $\theta_{T_o} w$ with the initial condition $\tilde{X}_{T_o}(x_o, w)$. By pathwise uniqueness, we see that $\tilde{X}_{T_o+t}(x_o, w) = X_{T_o+t}(x_o, w)$ almost surely for all $t \in [0, T_o]$. This means that $\tilde{X}_t(x_o, w)$ is a continuous version of $X_t(x_o, w)$ over $[0, 2T_o] \times \mathbf{R}^d$. Continuing in this way, we get a continuous version on the whole space $[0, +\infty[\times \mathbf{R}^d$.

**Step 2:** general case.

For $R > 0$, let $f_R(x)$ denote a smooth function with compact support satisfying

$$f_R(x) = 1 \quad \text{for } |x| \leq R \quad \text{and} \quad f_R(x) = 0 \quad \text{for } |x| > R + 1. \tag{2.9}$$

Define

$$\sigma_R(x) = \sigma(x) f_R(x) \quad \text{and} \quad f_R(x) = b(x) f_R(x). \tag{2.10}$$

Let $X_t^R(x,w)$ be the unique solution of the s.d.e. (0.1) with $\sigma$ and $b$ replaced by $\sigma_R$ and $b_R$. Let $\tilde{X}_t^R(x,w)$ denote a continuous version of $X_t^R(x,w)$. Such a version exists according to step 1. For $K > 0$, set

$$\tau_K^R(x) = \inf\{t > 0;\ |\tilde{X}_t^R(x,w)| \geq K\} \tag{2.11}$$

$$\tau_K(x) = \inf\{t > 0;\ |X_t(x,w)| \geq K\} \tag{2.12}$$

By the pathwise uniqueness, for $|x| \leq K$, we have

$$\tau_K(x) = \tau_K^{K+2}(x) \tag{2.13}$$

almost surely. For $|x| \leq R$, define

$$\tilde{X}_\cdot(x,w) = \tilde{X}_\cdot^{R+2}(x,w) \quad \text{on} \quad [0, \tau_R^{R+2}(x)) \tag{2.14}$$

Then it is clear that $\tilde{X}_\cdot(x,w)$ is a version of $X_\cdot(x,w)$. Let us prove that $\tilde{X}_t(x,w)$ is continuous in $(t,x)$ for almost all $w$. Fix $x_0$ with $|x_0| \leq K$. Since the life time of the solution is infinity, there exists $R > 0$ such that $\tau_R^{R+2}(x_0) > t$. This implies that $\sup_{0\leq s\leq t} |\tilde{X}_s^{R+2}(x_o,w)| < R$. By the continuity, we can find a neighbourhood $B_\delta(x_o)$ of $x_o$ such that $\sup_{0\leq s\leq t} |\tilde{X}_s^{R+2}(x,w)| < R$ or $\tau_R^{R+2}(x) > t$ for all $x \in B_\delta(x_o)$. Hence, $\tilde{X}_s(x,w) = \tilde{X}_s^{R+2}(x,w)$ for all $x \in B_\delta(x_o)$ and $s \leq t$, which implies that $\tilde{X}_s(x_o,w)$ is continuous with respect to $(s,x_o)$. ∎

**Remark 2.4:** Consider $r(s) = \log\frac{1}{s} \cdot \log\log\frac{1}{s}$ for $s \in ]0,1/2e]$. Clearly $s \to r(s)$ is decreasing and $s \to sr(s)$ is concave over $]0,1/2e]$. Applying (2.7), we get

$$\varphi(t) \leq \exp\Big(-\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}}\Big).$$

In order to apply the Kolmogorov's modification theorem, we have to find $\alpha > 0$ such that

$$\exp\Big(-\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}}\Big) \leq \varphi(0)^\alpha,$$

or

$$\big[\log\frac{1}{\varphi(0)}\big]^{e^{-C_p t}} \geq \alpha \log\frac{1}{\varphi(0)}$$

which is impossible when $|x_o - y_o|$ is small for any $t > 0$. ∎

## 3. Flow of homeomorphisms

**Theorem 3.1** *Assume that for* $|x-y| \le \frac{1}{2}$

$$\begin{cases} ||\sigma(x)-\sigma(y)||^2 & \le & C\,|x-y|^2 \,\log\frac{1}{|x-y|}, \\ |b(x)-b(y)| & \le & C\,|x-y|\,\log\frac{1}{|x-y|}, \end{cases}$$

*and* $\sigma$ *and* $b$ *are compactly supported. Then the solution of the s.d.e* (0.1) *admits a version* $X_t(x_o,w)$ *such that* $x_o \to X_t(x_o,w)$ *is a homeomorphism of* $\mathbf{R}^d$ *almost surely for all* $t>0$.

**Proof.** Let $R>0$ such that

$$\sigma(x)=0 \quad \text{and} \quad b(x)=0 \quad \text{for} \quad |x|\ge R.$$

By section 2, the solution of s.d.e. (0.1) admits a continuous version, still denoted by $X_t(x_o,w)$, in $(t,x_o)$, such that $|X_t(x_o,w)| \le R+1$ if $|x_o|\le R+1$. So $x_o \to \tilde{X}_t(x_o,w)$ defines a continuous map from $B(R+1)$ to $B(R+1)$, where

$$B(r)=\big\{x\in\mathbf{R}^d;\ |x|\le r\big\}.$$

**Lemma 3.2** *Let* $x_o \ne y_o$ *and* $\alpha<0$. *Then there exists* $C_\alpha, K_\alpha>0$ *such that*

$$\mathbb{E}\Big(|X_t(x_o)-X_t(y_o)|^{2\alpha}\Big) \le C_\alpha\,|x_o-y_o|^{2\alpha e^{-K_\alpha t}}, \quad \text{for } x_o, y_o \in B(R+1). \tag{3.1}$$

**Proof.** Let $0<\varepsilon<\dfrac{|x_o-y_o|^2}{8(R+1)^2}$. Consider $\eta_t(w)=X_t(x_o)-X_t(y_o)$ and

$$\xi_t(w)=\frac{|\eta_t(w)|^2}{8(R+1)^2}.$$

By theorem 1.1, we know that $\xi_t(w)\ne 0$ almost surely for all $t>0$. Define

$$\tau_\varepsilon=\inf\big\{t>0,\ \xi\le\varepsilon\big\}.$$

By Itô formula, for $s<t$,

$$\xi^\alpha_{t\wedge\tau_\varepsilon}-\xi^\alpha_{s\wedge\tau_\varepsilon}=\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon}\alpha\xi_u^{\alpha-1}\,d\xi_u+\frac{1}{2}\alpha(\alpha-1)\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon}\alpha\xi_u^{\alpha-2}\,d\xi_u\cdot d\xi_u.$$

Where

$$\begin{aligned} d\xi_t=\frac{1}{4(R+1)^2}\Big\{&\big\langle\eta_t(w),\big(\sigma(X_t(x_o))-\sigma(X_t(y_o))\big)dw_t\big\rangle \\ &+\big\langle\eta_t(w),b(X_t(x_o))-b(X_t(y_o))\big\rangle\,dt \\ &+\frac{1}{2}||\sigma(X_t(x_o))-\sigma(X_t(y_o))||^2\,dt\Big\}, \end{aligned}$$

and

$$d\xi_t \cdot d\xi_t = \Big(\frac{1}{4(R+1)^2}\Big)^2 |\big(\sigma^*(X_t(x_o)) - \sigma^*(X_t(y_o))\big)\eta_t|^2\, dt.$$

Using lemma 2.1 for $p = 1$, there exists a constant $C > 0$ such that

$$|b(X_t(x_o)) - b(X_t(y_o))| \le C\,|\eta_t| \log \frac{8(R+1)^2}{|\eta_t|^2} \tag{3.2}$$

and

$$||\sigma(X_t(x_o)) - \sigma(X_t(y_o))||^2 \le C\,|\eta_t|^2 \log \frac{8(R+1)^2}{|\eta_t|^2}. \tag{3.3}$$

Therefore by (3.2),

$$\begin{aligned}&\Big|\alpha\xi_u^{\alpha-1}\big\langle \eta_u, b(X_t(x_o)) - b(X_t(y_o))\big\rangle\Big|\\ &\le -2C\alpha\, \xi_u^\alpha \log\frac{1}{\xi_u} = -2C\, \xi_u^\alpha \log \frac{1}{\xi_u^\alpha}.\end{aligned}$$

Similarly , we have

$$\begin{aligned}&\frac{1}{2}\alpha(\alpha-1)\xi_u^{\alpha-2}\Big(\frac{1}{4(R+1)^2}\Big)^2 |\big(\sigma^*(X_t(x_o)) - \sigma^*(X_t(y_o))\big)\eta_t|^2\\ &\le 2C|\alpha-1|\, \xi_u^\alpha \log\frac{1}{\xi_u^\alpha}.\end{aligned}$$

Let $dM_t$ be the martingale part of $d\xi_t$. Then we have

$$\xi^\alpha_{t\wedge\tau_\varepsilon} - \xi^\alpha_{s\wedge\tau_\varepsilon} \le \int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon} \alpha\xi_u^{\alpha-1}\, dM_u + K(\alpha)\int_{s\wedge\tau_\varepsilon}^{t\wedge\tau_\varepsilon} \xi_u^\alpha \log\frac{1}{\xi_u^\alpha}\, du.$$

Define $\varphi(t) = I\!E(\xi^\alpha_{t\wedge\tau_\varepsilon})$. From the above inequality, we get

$$\varphi'(t) \le K(\alpha)\varphi(t)\log\frac{1}{\varphi(t)},$$

which implies that

$$\varphi(t) \le \varphi(0)^{e^{-K(\alpha)\,t}}.$$

Hence,

$$I\!E\big(\xi^\alpha_{t\wedge\tau_\varepsilon}\big) \le \Big(\frac{|x_o - y_o|^2}{8(R+1)^2}\Big)^{\alpha\, e^{-K(\alpha)\,t}} \qquad \text{for } x_o, y_o \in B(R+1).$$

Since $\tau_\varepsilon \uparrow +\infty$ while $\varepsilon \downarrow 0$, we get

$$I\!E\big(\xi^\alpha_t\big) \le \Big(\frac{|x_o - y_o|^2}{8(R+1)^2}\Big)^{\alpha\, e^{-K(\alpha)\,t}} \qquad \text{for } x_o, y_o \in B(R+1)$$

or (3.1) holds. ∎

**Lemma 3.3** *Let $\delta > 0$ and $\Delta_\delta = \big\{(x_o, y_o) \in B(R+1) \times B(R+1);\ |x_o - y_o| \geq \delta\big\}$. Take a continuous version $X_t(x_o)$ and set $\eta_t(x_o, y_o) = |X_t(x_o) - X_t(y_o)|^{-1}$. Then for any $p > 1$, there exist constant $C_{p,T,\delta}, K_1(p) > 0$ such that*

$$\begin{aligned} &\mathbb{E}\Big(|\eta_t(x_o, y_o) - \eta_s(\tilde{x}_o, \tilde{y}_o)|^p\Big) \\ &\leq C_{p,T,\delta}\Big(|x_o - \tilde{x}_o|^{pe^{-K_1(p)T}} + |y_o - \tilde{y}_o|^{pe^{-K_1(p)T}} + |t - s|^{p/2}\Big) \end{aligned} \tag{3.4}$$

*for $(x_o, y_o), (\tilde{x}_o, \tilde{y}_o) \in \Delta_\delta$ and $s, t \in [0, T]$.*

**Proof.** We have

$$\begin{aligned} &|\eta_t(x_o, y_o) - \eta_s(\tilde{x}_o, \tilde{y}_o)| \\ &= \eta_t(x_o, y_o)\eta_s(\tilde{x}_o, \tilde{y}_o)\Big|\,|X_t(x_o) - X_t(y_o)| - |X_s(\tilde{x}_o) - X_s(\tilde{y}_o)|\,\Big| \\ &\leq \eta_t(x_o, y_o)\eta_s(\tilde{x}_o, \tilde{y}_o)\Big(|X_t(x_o) - X_s(\tilde{x}_o)| + |X_t(y_o) - X_s(\tilde{y}_o)|\Big). \end{aligned}$$

By lemma 3.1, there exists a constant $C_{p,T,\delta} > 0$ such that

$$\mathbb{E}\big(\eta_t(x_o, y_o)^{4p}\big) \leq C_{p,T,\delta} \quad \text{for } (x_o, y_o) \in \Delta_\delta,\ t \in [0, T].$$

Now combining with (2.8), we get

$$\begin{aligned} &\mathbb{E}\Big(|\eta_t(x_o, y_o) - \eta_s(\tilde{x}_o, \tilde{y}_o)|^p\Big) \\ &\leq C_{p,T,\delta}\Big(|x_o - \tilde{x}_o|^{2pe^{-K_1(p)T}} + |y_o - \tilde{y}_o|^{2pe^{-K_1(p)T}} + 2|t - s|^p\Big)^{1/2} \end{aligned}$$

which is dominated by the right hand side of (3.4). So we get the result. ∎

**Proof of theorem 3.1** By (3.4), for $p > 2(2d+1)$ and $T_o > 0$ small enough, we can apply the Kolmogorov's modification theorem to get that $\eta_t(x_o, y_o)$ has a continuous version $\tilde{\eta}_t(x_o, y_o)$ on $[0, T_o] \times \Delta_o$. This means that $(t, x_o, y_o) \to \tilde{\eta}_t(x_o, y_o)$ is continuous on $[0, T_o] \times \Delta_o$ almost surely. Let $D$ be a countable dense subset of $[0, T_o] \times \Delta_o$. Then almost surely, for all $(t, x_o, y_o) \in D$,

$$\tilde{\eta}_t(x_o, y_o) = |X_t(x_o) - X_t(y_o)|^{-1},$$

or

$$|X_t(x_o) - X_t(y_o)| = \tilde{\eta}_t(x_o, y_o)^{-1}. \tag{3.5}$$

Now by continuity, the relation (3.5) holds for all $(t, x_o, y_o) \in [0, T_o] \times \Delta_o$. In particular, this relation shows that almost surely for all $t > 0$, $x_o \to X_t(x_o)$ is injective on $B(R+1)$. Since $X_t(x_o) = x_o$ for $|x_o| > R$, $x_o \to X_t(x_o)$ is injective on the whole $\mathbf{R}^d$. Let $\overline{\mathbf{R}^d} = \mathbf{R}^d \cup \{\infty\}$

which is homeomorph to the sphere $S^d$. Extend the map $X_t$ to $\overline{\mathbf{R}^d}$ by setting $X_t(\infty) = \infty$. It is clear that $X_t$ is continuous near $\infty$. So $X_t$ can be seen as a continuous map from $S^d$ into $S^d$. Since $X_t$ is homotope to the identity, we conclude that $X_t$ is a surjective map on $S^d$. Therefore $X_t$ is a homeomorphism of $S^d$ and its restriction $X_t$ on $\mathbf{R}^d$ is a homeomorphism of $\mathbf{R}^d$. Now using the relation

$$X_{T_o+t}(x_o, w) = X_t\big(X_{T_o}(x_o, w), \theta_{T_o} w\big),$$

we conclude that $X_t$ is a homeomorphsim of $\mathbf{R}^d$ for all $t > 0$. ∎

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